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DERIVATION OF EQUATIONS FOR CONVERTING FROM GEODETIC COORDINATES TO GEOCENTRIC COORDINATES

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DERIVATION OF EQUATIONS FOR CONVERTING FROM GEODETIC COORDINATES TO GEOCENTRIC COORDINATES

F. T. Heuring

In the A.P.L. orbit computation programs, the TRANET Tracking Stations are specified in a geocentric coordinate system, whereas, particular positions (such as a TRANET Tracking Site) over the Earth are expressed initially in a geodetic coordinate system. In order to acquire geocentric coordinates from a given set of geodetic coordinates a set of transformation equations were derived.

Section I will define the notation, and Section II will embody the derivation of the transformation equations.

I. Notation*

Let:

φ_{G_i} = geodetic latitude of i-th tracking site in its local datum,

λ_{G_i} = geodetic longitude of i-th tracking site in its local datum,

h_i = elevation of i-th tracking site above (below) geoid,

H_i = geoidal height of i-th tracking site in its local datum,

ξ_i = deflection in meridian at i-th tracking site,

η_i = deflection in prime vertical at i-th tracking site,

a_i = equatorial radius of the local datum spheroid of the i-th tracking site, scaled by R_o,

b_i = polar radius of the local datum spheroid on the i-th tracking site, scaled by R_o,

*See References 1 and 2 for definition of geodetic, datum, etc.

$x_{G_i}, y_{G_i}, z_{G_i}$ = cartesian coordinates, scaled by R_o, on i-th datum spheroid as specified by tracking site φ_G and λ_G, (cartesian origin identical to i-th datum origin),

$x_{H_i}, y_{H_i}, z_{H_i}$ = cartesian coordinates, scaled by R_o, on geoid as specified by tracking site H, (cartesian origin identical to i-th datum origin),

$x_{E_i}, y_{E_i}, z_{E_i}$ = cartesian coordinates, scaled by R_o, of tracking site on earth's surface, (cartesian origin identical to i-th datum origin),

$\Delta x_i, \Delta y_i, \Delta z_i$ = center of spheroid of the i-th tracking site datum in the A.P.L. Datum, scaled by R_o,

$\zeta_{G_i} = (x_{G_i}^2 + y_{G_i}^2)^{\frac{1}{2}}$

R_o = equatorial radius of A.P.L. Datum spheroid,

$x_{c_i}, y_{c_i}, z_{c_i}$ = cartesian coordinates of tracking site in A.P.L. geocentric coordinates, scaled by R_o,

r_{c_i} = radius of i-th tracking site in A.P.L. geocentric coordinates, scaled by R_o,

φ_{c_i} = latitude of i-th tracking site in A.P.L. geocentric coordinates,

λ_{c_i} = longitude of i-th tracking site in A.P.L. geocentric coordinates,

$\zeta_{c_i} = (x_{c_i}^2 + y_{c_i}^2)^{\frac{1}{2}}$.

II. Derivation

A. Given φ_{G_i}, λ_{G_i}, a_i and b_i, conversion to x_{G_i}, y_{G_i}, z_{G_i} and ζ_{G_i} is as follows. Using the equation for an ellipse

$$\frac{\zeta_{G_i}^2}{a_i^2} + \frac{z_{G_i}^2}{b_i^2} = 1,$$

in particular the ellipse is a meridional plane of the i-th datum; differentiate z_{G_i} with respect to ζ_{G_i}

$$\frac{\partial z_{G_i}}{\partial \zeta_{G_i}} = - \frac{b_i^2}{a_i^2} \frac{\zeta_{G_i}}{z_{G_i}} .$$

But (see Figure 1A),

$$\frac{\partial z_{G_i}}{\partial \zeta_{G_i}} = - \frac{1}{\tan \varphi_{G_i}}$$

from which by algebraic manipulation (Figure 1B),

$$\zeta_{G_i} = \frac{a_i}{\left(1 + \left(\frac{b_i}{a_i}\right)^2 \tan^2 \varphi_{G_i}\right)^{\frac{1}{2}}} ,$$

after which,

$$\left.\begin{aligned} x_{G_i} &= \zeta_{G_i} \cos \lambda_{G_i} \\ y_{G_i} &= \zeta_{G_i} \sin \lambda_{G_i} \\ z_{G_i} &= \zeta_{G_i} \frac{b_i^2}{a_i^2} \tan \varphi_{G_i} . \end{aligned}\right\} \quad (1)$$

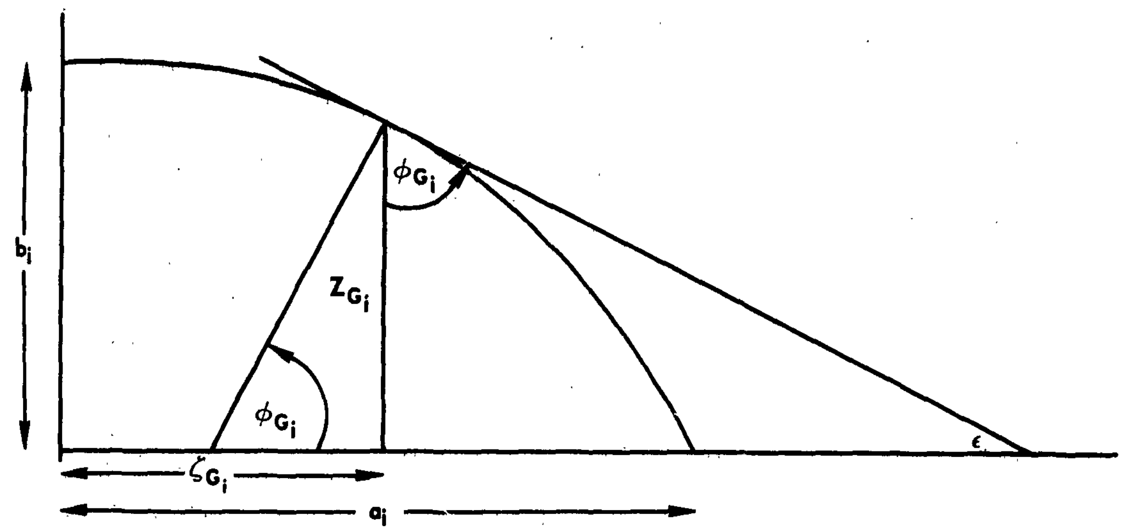

Figure 1A Meridian Plane in i-th Datum

$$\frac{1}{\tan \phi_{G_i}} = \cot \phi_{G_i} = \tan \epsilon - \frac{\partial z_{G_i}}{\partial \zeta_{G_i}}$$

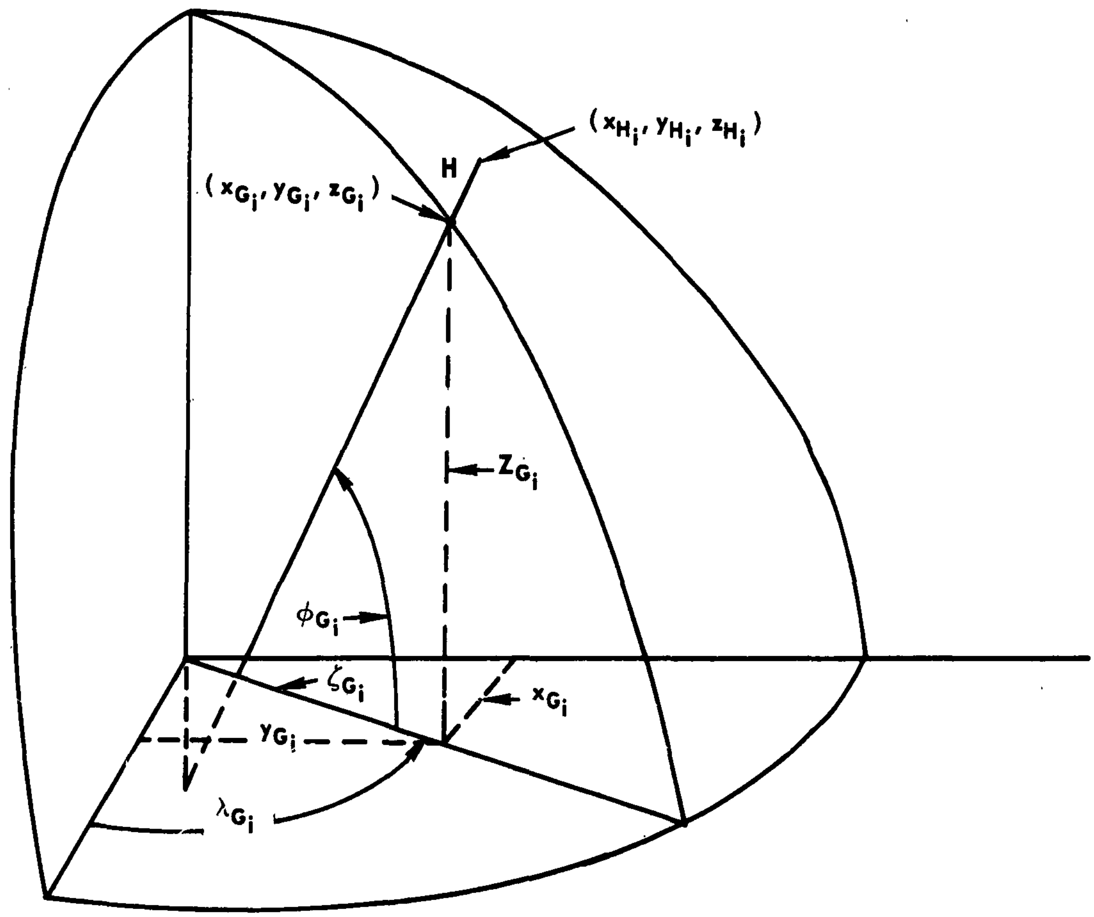

Figure 1B

Pictorial view of geodetic ($\phi_{G_i}, \lambda_{G_i}$), cartesian "geodetic" ($x_{G_i}, y_{G_i}, z_{G_i}$) and cartesian "geoidal" ($x_{H_i}, y_{H_i}, z_{H_i}$) coordinates.

B. Compute x_{H_i}, y_{H_i}, z_{H_i} (Figure 1B). H_i is an extension of the normal to the spheroid, consequently,

$$
\begin{aligned}
x_{H_i} &= x_{G_i} + H_i \cos \varphi_{G_i} \cos \lambda_{G_i} \\
y_{H_i} &= y_{G_i} + H_i \cos \varphi_{G_i} \sin \lambda_{G_i} \\
z_{H_i} &= z_{G_i} + H_i \sin \varphi_{G_i} .
\end{aligned}
\tag{2}
$$

C. Compute x_{E_i}, y_{E_i}, z_{E_i} by considering h_i, ξ_i and η_i (see Figure 2).

$$
\begin{aligned}
x_{E_i} &= x_{H_i} + h_i \cos (\varphi_{G_i} + \xi_i) \cos (\lambda_{G_i} + \Delta\lambda_i) \\
y_{E_i} &= y_{H_i} + h_i \cos (\varphi_{G_i} + \xi_i) \sin (\lambda_{G_i} + \Delta\lambda_i) \\
z_{E_i} &= z_{H_i} + h_i \sin (\varphi_{G_i} + \xi_i).
\end{aligned}
\tag{3}
$$

From law of cosines for spherical triangles (Figure 2), $\Delta\lambda_i$ can be approximated.

$$\cos \Delta\lambda_i = \frac{\cos \eta_i - \sin^2 (\varphi_{G_i} + \xi_i)}{\cos^2 (\varphi_{G_i} + \xi_i)}. \tag{4}$$

(Restrict $\Delta\lambda_i$ to have the same sign as η_i).

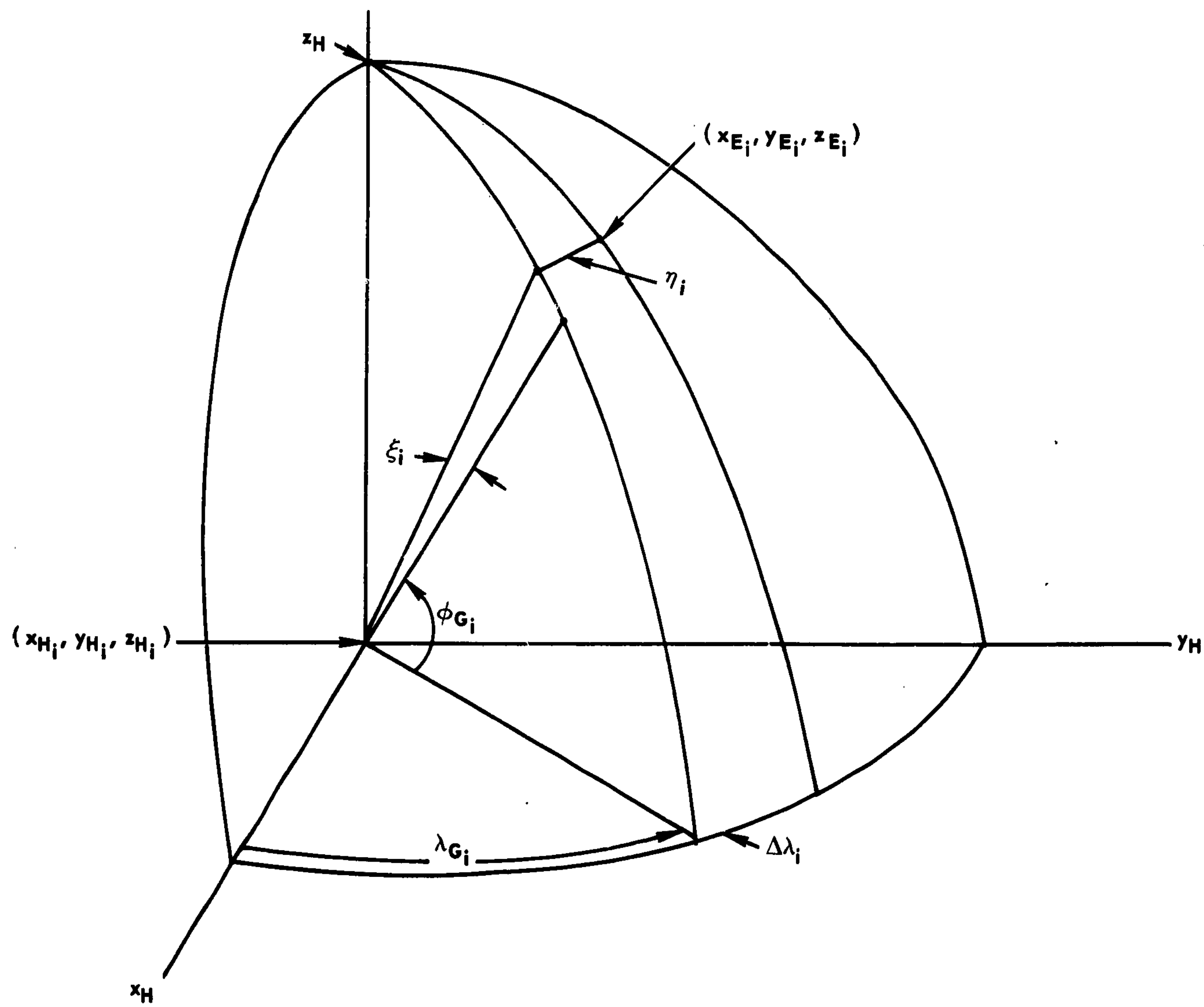

Figure 2
Diagram of the deflections of the vertical (ξ_i and η_i) and the associated quantities necessary to acquire the cartesian coordinates on the geoid from earth surface cartesian coordinates.

D. Let us simplify by expanding small quantities. Assume:

$$\xi_i, \eta_i \leqq 30''^{*} \text{ (of arc)};$$

and

$$1^\circ < |\varphi_{G_i}| < 89^\circ;$$

and only take quantities of magnitude ξ_i, η_i and $\Delta\lambda_i$ to second order.

$$\cos \xi_i \doteq 1 - \frac{\xi_i^2}{2}, \ \sin \xi_i \doteq \xi_i$$

$$\cos \eta_i \doteq 1 - \frac{\eta_i^2}{2}, \ \sin \eta_i \doteq \eta_i$$

$$\cos \Delta\lambda_i \doteq 1 - \frac{\Delta\lambda_i^2}{2}$$

thus,

$$\cos^2 (\varphi_{G_i} + \xi_i) = [\cos \varphi_{G_i} (1 - \frac{\xi_i^2}{2}) - \xi_i \sin \varphi_{G_i}]^2$$

$$= (1 - \frac{\xi_i^2}{2})^2 \cos^2 \varphi_{G_i} + \xi_i^2 \sin^2 \varphi_{G_i}$$

$$- 2 \xi_i (1 - \frac{\xi_i^2}{2}) \sin \varphi_{G_i} \cos \varphi_{G_i}$$

$$= \cos^2 \varphi_{G_i} - \xi_i \sin 2\varphi_{G_i} - \xi_i^2 \cos 2\varphi_{G_i} + \text{3rd order} \qquad (5)$$

*From a personal communication with Mr. L. Simmons, U.S.C. and G.S., deflection of 30" exist but are in general uncommon.

$$\sin^2(\varphi_{G_i} + \xi_i) \doteq \left[\sin\varphi_{G_i}\left(1 - \frac{\xi_i^2}{2}\right) + \xi_i \cos\varphi_{G_i}\right]^2$$

$$\doteq \left(1 - \frac{\xi_i^2}{2}\right)\sin^2\varphi_{G_i} + \xi_i^2\cos^2\varphi_{G_i} + 2\,\xi_i \sin\varphi_{G_i}\cos\varphi_{G_i}$$

$$= \sin^2\varphi_{G_i} + \xi_i \sin 2\varphi_{G_i} + \xi_i^2 \cos 2\varphi_{G_i} + \text{3rd order} \tag{6}$$

$$\cos(\varphi_{G_i} + \xi_i) = \cos\varphi_{G_i}\left(1 - \frac{\xi_i^2}{2}\right) - \xi_i \sin\varphi_{G_i} = \cos\varphi_{G_i} - \xi_i\sin\varphi_{G_i} - \frac{\xi_i^2}{2}\cos\varphi_{G_i} \tag{7}$$

$$\sin(\varphi_{G_i} + \xi_i) = \sin\varphi_{G_i}\left(1 - \frac{\xi_i^2}{2}\right) + \xi_i \cos\varphi_{G_i} = \sin\varphi_{G_i} + \xi_i\cos\varphi_{G_i} - \frac{\xi_i^2}{2}\sin\varphi_{G_i} \tag{8}$$

from equation (4):

$$1 - \frac{\eta_i^2}{2} = \sin^2(\varphi_{G_i} + \xi_i) + \cos^2(\varphi_{G_i} + \xi_i)\left[1 - \frac{\Delta\lambda_i^2}{2}\right],$$

$$= 1 - \frac{\Delta\lambda_i^2}{2}\cos^2(\varphi_{G_i} + \xi_i),$$

$$\Delta\lambda_i = \frac{\eta_i}{\cos(\varphi_{G_i} + \xi_i)},$$

and using equation (7),

$$\Delta\lambda_i = \frac{\eta_i}{\cos\varphi_{G_i}}\left[\frac{1}{1 - \xi_i \tan\varphi_{G_i} - \frac{\xi_1^2}{2}}\right] = \frac{\eta_i}{\cos\varphi_{G_i}}\left[1 + \xi_i \tan\varphi_{G_i} + \frac{\xi_i^2}{2} + \frac{\xi_i^2}{2}\tan^2\varphi_{G_i}\right]$$

$$= \eta_i \sec\varphi_{G_i}\left[1 + \xi_i \tan\varphi_{G_i}\right] + \text{3rd order.} \qquad (9)$$

Further,

$$\sin(\lambda_{G_i} + \Delta\lambda_i) = \sin\lambda_{G_i}\left(1 - \frac{\eta_i^2 \sec^2\varphi_{G_i}}{2}\right) + \cos\varphi_{G_i}\,\eta_i \sec\varphi_{G_i}\,(1 + \xi_i \tan\varphi_{G_i})$$

$$= \sin\lambda_{G_i} + \eta_i \frac{\cos\lambda_{G_i}}{\cos\varphi_{G_i}} + \frac{\eta_i}{\cos^2\varphi_{G_i}}\left[\xi_i \cos\lambda_{G_i} \sin\varphi_{G_i} - \frac{\eta_i}{2}\sin\lambda_{G_i}\right] + \text{3rd order} \qquad (10)$$

$$\cos(\lambda_{G_i} + \Delta\lambda_i) = \cos\lambda_{G_i} - \sin\lambda_{G_i} \sec\varphi_{G_i}\,(1 + \xi_i \tan\varphi_{G_i})\,\eta_i - \frac{\eta_i^2}{2}\sec^2\varphi_{G_i}\cos\lambda_{G_i}$$

$$= \cos\lambda_{G_i} - \eta_i \frac{\sin\lambda_{G_i}}{\cos\varphi_{G_i}} - \frac{\eta_i}{\cos^2\varphi_{G_i}}\left[\xi_i \sin\lambda_{G_i} \sin\varphi_{G_i} + \frac{\eta_i}{2}\cos\lambda_{G_i}\right] + \text{3rd order.} \qquad (11)$$

E. Using equations (1), (2), (3), (7), (8), (9), (10), and (11), x_{E_i}, y_{E_i} and z_{E_i} can be expressed as functions of the geodetic inputs (φ_{G_i}, λ_{G_i}, h_i, H_i, η_i, ξ_i, a_i, and b_i).

$$\left.\begin{aligned}
x_{E_i} &= \zeta_{G_i} \cos\lambda_{G_i} + H_i \cos\varphi_{G_i} \cos\lambda_{G_i} + h_i (\cos\varphi_{G_i} - \xi_i \sin\varphi_{G_i}) \left(\cos\lambda_{G_i} - \eta_i \frac{\sin\lambda_{G_i}}{\cos\varphi_{G_i}}\right) \\
&= \zeta_{G_i} \cos\lambda_{G_i} + (H_i + h_i) \cos\varphi_{G_i} \cos\lambda_{G_i} - h_i(\xi_i \sin\varphi_{G_i} \cos\lambda_{G_i} + \eta_i \sin\lambda_{G_i}) + \text{3rd order.} \\
y_{E_i} &= \zeta_{G_i} \sin\lambda_{G_i} + H_i \cos\varphi_{G_i} \sin\lambda_{G_i} + h_i(\cos\varphi_{G_i} - \xi_i \sin\varphi_{G_i}) \left(\sin\lambda_{G_i} + \eta_i \frac{\cos\lambda_{G_i}}{\cos\varphi_{G_i}}\right) \\
&= \zeta_{G_i} \sin\lambda_{G_i} + (H_i + h_i) \cos\varphi_{G_i} \sin\lambda_{G_i} - h_i(\xi_i \sin\varphi_{G_i} \sin\lambda_{G_i} - \eta_i \cos\lambda_{G_i}) + \text{3rd order.} \\
z_{E_i} &= \zeta_{G_i} \frac{b_i^2}{a_i^2} \tan\varphi_{G_i} + H_i \sin\varphi_{G_i} + h_i(\sin\varphi_{G_i} + \xi_i \cos\varphi_{G_i}) \\
&= \zeta_{G_i} \frac{b_i^2}{a_i^2} \tan\varphi_{G_i} + (H_i + h_i) \sin\varphi_{G_i} + h_i \xi_i \cos\varphi_{G_i} + \text{3rd order.}
\end{aligned}\right\} \quad (12)$$

F. The cartesian coordinates in the A.P.L. geocentric system are:

$$x_{c_i} = x_{E_i} + \Delta x_i$$

$$y_{c_i} = y_{E_i} + \Delta y_i$$

$$z_{c_i} = z_{E_i} + \Delta z_i$$

where Δx_i, Δy_i, and Δz_i are of second order, at best.

H. The cylindrical coordinates $(z_{c_i}, \zeta_{c_i}, \lambda_{c_i})$ in the A.P.L. geocentric system are:

$$z_{c_i} = \zeta_{G_i} \frac{b_i^2}{2} \tan \varphi_{G_i} + (H_i + h_i) \sin \varphi_{G_i} + h_i \xi_i \cos \varphi_{G_i} + \Delta z_i + \text{3rd order} \tag{13}$$

$$\zeta_{c_i}^2 = x_{c_i}^2 + y_{c_i}^2$$

After some algebraic manipulation and using the binominal expansion

$$\begin{aligned} \zeta_{c_i} = {} & \zeta_{G_i} + (H_i + h_i) \cos \varphi_{G_i} + \Delta x_i \cos \lambda_{G_i} + \Delta y_i \sin \lambda_{G_i} - h_i \xi_i \sin \varphi_{G_i} \\ & + (H_i + h_i) \cos \varphi_{G_i} \cdot \frac{1}{\zeta_{G_i}} (\Delta x_i \cos \lambda_{G_i} + \Delta y_i \sin \lambda_{G_i}) \\ & + \text{3rd order.} \end{aligned} \tag{14}$$

In the derivation of λ_{G_i}, no previously derived quantities were used as was the case with ζ_{c_i}. From Figure 3A, h_i is considered to be zero, thus the angle σ can be approximated as follows:

$$\epsilon_1 + \epsilon_2 = \Delta x_i \sin \lambda_{G_i}$$

$$\epsilon_2 \doteq \Delta y_i \cos \lambda_{G_i} \qquad \text{where } \epsilon_1 \text{ and } \epsilon_2 \text{ are normal to } \zeta_{G_i}\text{, and}$$

$$\epsilon_1 \doteq \Delta x_i \sin \lambda_{G_i} - \Delta y_i \cos \lambda_{G_i}.$$

Since ϵ_1 considered, at best, second order,

$$\sigma \doteq \frac{\epsilon_1}{\zeta_{G_i}}$$

and from the geometry,

$$\lambda_{c_i} = \lambda_{G_i} - \sigma = \lambda_{G_i} - \frac{1}{\zeta_{G_i}} (\Delta x_i \sin \lambda_{G_i} - \Delta y_i \cos \lambda_{G_i}) \tag{15}$$

Upon including the station elevation (h_i) and deflection in the prime vertical (η_i) (see Figure 3B)

$$\tau = h_i \sin \eta_i = h_i \eta_i \quad (\eta_i \text{ is of magnitude } \pm 30'' \text{ of arc})$$

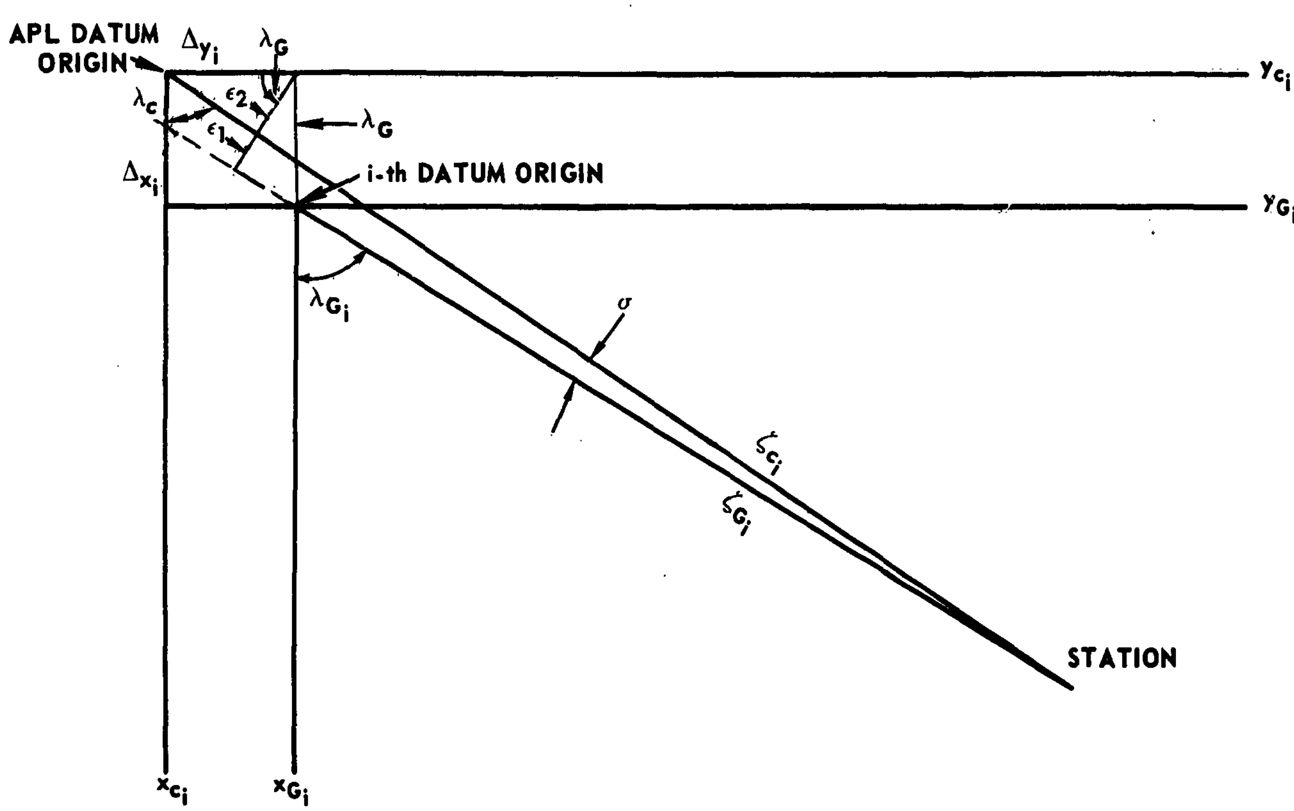

Figure 3A Diagram showing means of determining λ_c when $h_i = 0$.

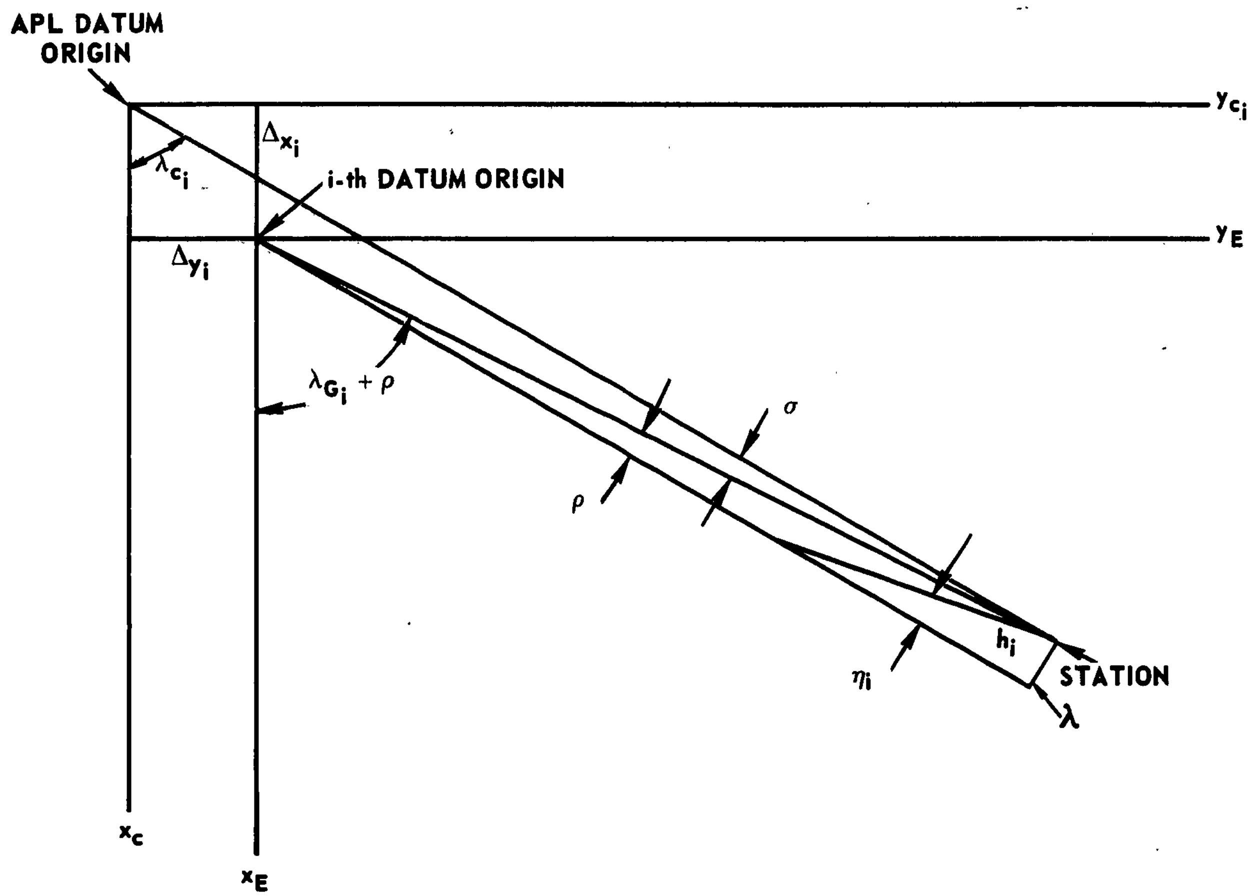

Figure 3B Diagram showing determination of λ_c when $h_i \neq 0$.

and it follows similarly

$$\rho \doteq \frac{\tau}{\zeta_{G_i}} = \frac{h_i \, \eta_i}{\zeta_{G_i}}$$

From equation (15) and Figure 3B,

$$\lambda_{c_i} = \lambda_{G_i} + \rho - \sigma$$

$$= \lambda_{G_i} + \frac{h_i \, \eta_i}{\zeta_{G_i}} - \frac{1}{\zeta_{G_i}} \left[\Delta x_i \sin (\lambda_{G_i} + \rho) - \Delta y_i \cos (\lambda_{G_i} + \rho) \right]$$

Assuming $\cos \rho = 1 - \frac{\rho^2}{2}$, $\sin \rho = \rho$,

$$\lambda_{c_i} = \lambda_{G_i} + \frac{1}{\zeta_{G_i}} \left[h_i \, \eta_i - (\Delta x_i \sin \lambda_{G_i} - \Delta y_i \cos \lambda_{G_i}) \right] + \text{3rd order.} \qquad (16)$$

Equations (13), (14), and (16) are the cylindrical coordinates z_{c_i}, ζ_{c_i}, λ_{c_i} in the A.P.L. Earth fixed coordinate system expressed as a function of the geodetic coordinates of a tracking station.

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1. Bomford, Brigadier G., "Geodesy", Clarendon Press, 1952.

2. Hasner, George L., "Geodesy", Wiley, Second Edit., 1930, (Chap. V - Properties of the Spheroid and Chap. VIII - Figure of the Earth).

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